Non-Hermitian random matrix models: Free random variable approach

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Using the standard concepts of free random variables, we show that for a large class of non-Hermitian random matrix models, the support of the eigenvalue distribution follows from their Hermitian analogs using a conformal transformation. We also extend the concepts of free random variables to the class of non-Hermitian matrices, and apply them to the models discussed by Ginibre-Girko (elliptic ensemble) [J. Ginibre, J. Math. Phys. **6**, 1440 (1965); V. L. Girko, *Spectral Theory of Random Matrices* (in Russian) (Nauka, Moscow, 1988)] and Mahaux-Weidenmüller (chaotic resonance scattering) [C. Mahaux and H. A. Weidenmüller, *Shell-model Approach to Nuclear Reactions* (North-Holland, Amsterdam, 1969)]. [S1063-651X(97)09704-3]

I. INTRODUCTION

The distribution of eigenvalues of complex and large random matrices is of relevance to a variety of physical problems. Non-Hermitian random matrices appear naturally in the evolution of dissipative quantum many-body systems [1], in quantum chaotic scattering [2], in quantum optics [3], and possibly in quantum chromodynamics [4].

The distribution of complex energies of unstable quantum systems in the framework of random matrix models, has been investigated in some details by Sokolov and Zelevinsky [5], following on the original work of Weidenmüller and co-workers [1,6]. Alternative and extended analyses can be also found in [7] using the replica method, and in [8] using the supersymmetric method. For large matrices, a structural change in the eigenvalue distribution within the complex plane was observed in the case of strong non-Hermiticity. The level density has been used to assess the statistical distribution of the resonance widths in chaotic scattering with large scattering channels [7], and the time delay in chaotic scattering [9,10]. The latter may be of relevance for a quantitative assessment of quantum chaos [2]. Weak non-Hermitian ensembles were recently investigated in [11].

In this paper, we would like to show how to evaluate, in a straightforward way, the supports for the level density, as well as the eigenvalues distribution for a large class of non-Hermitian and random matrices using the concepts of free random variables as developed by Voiculescu, Dykema, and Nica [12] and popularized by Zee [13]. In Sec. II, we outline the general definitions for the standard S and R transformations, as known for Hermitian random matrices. In Sec. III, we discuss the resonance matrix model as an example of a non-Hermitian random matrix model. In Sec. IV using the concepts of S and R transforms we derive the spectral density of its Hermitian analog. In Sec. V, we show how the support for a non-Hermitian spectral density follows from the Hermitian one using a conformal transformation. Our arguments are applied to Ginibre-Girko's [14] and Mahaux-Weidenmüller's [15] random matrix models. In Sec. VI, we generalize the concept of R transforms to the non-Hermitian case, and use it to analyze the resolvent of non-Hermitian random matrix models. Our conclusions are summarized in Sec. VII.

II. R AND S TRANSFORMATIONS

Addition and multiplication of free random variables can be assessed using R and S transformations [12]. Specifically, the R transformation is additive $R_{1+2}=R_1+R_2$, and the S transformation is multiplicative $S_{1\times2}=S_1S_2$.

For the sum, if we were to define a Blue's function $B(z)$ [13] as the functional inverse of a Green's function $G(z)$, that is $B[G(z)]=G[B(z)]=z$, then the R function is simply $R(z)=B(z)+1/z$. Physically, the R transform is some pertinent self-energy in the planar approximation [13]. The additive property of the R transform implies the addition law [12,13],

$$B_{1+2}(z)=B_1(z)+B_2(z)-\frac{1}{z} \tag{1}$$

for the Blue's functions. Hence, the problem of finding the spectral distribution of the sum of two free random matrices is linear, and follows from the simple algorithm: Find G_1 and G_2, construct their functional inverses B_1 and B_2, add them through Eq. (1), and invert B_{1+2} to get G_{1+2}. The spectral density of the sum is the discontinuity of G_{1+2} along the real axis. The zeroes of $B'(z)=0$ characterize the end points of the spectral density [13], and reflect structural changes in the underlying spectrum.

For the product, if we were to define $\chi(z)$ as a solution of [12]

$$\frac{1}{\chi(z)}G\left(\frac{1}{\chi(z)}\right)-1=z \qquad (2)$$

then the S function is simply

$$S(z)=\frac{1+z}{z}\chi(z). \qquad (3)$$

The S transform of the product of two free random matrices is the product of their S transforms that is $S_{1\times2}=S_1 S_2$. Given $S_{1\times2}$, the resolvent $G_{1\times2}$ follows through Eqs. (2) and (3) in reverse order.

III. RESONANCE SCATTERING MODEL

To make part of our subsequent discussions clear, we will use the non-Hermitian random matrix model introduced by Mahaux and Weidenmüller for resonance scattering [15] as inspired by the Weisskopf and Wigner effective Hamiltonian [16], to illustrate some of our assertions. In brief, a quantum system composed of $(N-M)$ closed and M open channels can be described by the effective scattering Hamiltonian

$$H=H_R+ig\Gamma, \quad \Gamma=AA^T, \qquad (4)$$

which is $N\times N$ dimensional. A is an asymmetric $N\times M$ random matrix, and g an overall coupling. Unitarity enforces the form of Γ used in Eq. (4). Wigner's condition implies $g<0$. The matrix elements A_k^a characterize the transition between the $(N-M)$-internal channels and M-external channels, and are assumed to be independent of the scattering energy. For $M=0$, the Hamiltonian is real and the spectrum is bound. For $M\neq0$, the Hamiltonian is complex, with all states acquiring a width.

The resolvent $G(z)$ associated to Eq. (4) is defined as

$$G(z)=\frac{1}{N}\left\langle\mathrm{Tr}\frac{1}{z-H}\right\rangle, \qquad (5)$$

with complex valued poles reflecting the energy and width of the resonance states. Its imaginary part is related to the total lifetime (time delay) in the scattering process. The averaging in Eq. (5) corresponds to the GOE ensemble for H_R, with the transition matrix elements A_k^a treated as independent Gaussian variables [7]. The simpler case (''f case'' [6,8]), where the matrix elements are fixed, will be briefly discussed later. In its domain of analyticity D, $G(z)$ is a holomorphic function of z. In the complement $\bar{D}$, $G(z)$ is in general non-holomorphic, with a nonvanishing spectral distribution [17]

$$\nu(z,\bar{z})=\frac{1}{\pi}\frac{\partial G(z)}{\partial\bar{z}}. \qquad (6)$$

The latter provides for a statistical analysis of the characteristics of the resonances in chaotic scattering for a large number of channels [7,9,18]. It can also be used to discuss Dicke super-radiance in quantum optics [3].

IV. HERMITIAN CASE

First consider the case of an $N\times N$ real symmetric product matrix $\Gamma_{kl}^S=A_k^a A_l^a$. In the large N limit, the spectral function is connected to the one of the random $N\times N$ matrix A as

$$\nu_\Gamma^{N\times N}(z=\lambda^2)=\frac{\nu_A(\lambda)}{|\lambda|}=\frac{1}{2\pi}\frac{\sqrt{4-z}}{\sqrt{z}}. \qquad (7)$$

The numerator of Eq. (7) is just Wigner's semicircular law [19]. The case of rectangular $N\times M$ matrices with $N,M\to\infty$ but $m=M/N$ fixed, follows by truncation using the projector [20]

$$P=\mathrm{diag}(\underbrace{1,\ldots,1}_{M},\underbrace{0,\ldots,0}_{N-M}) \qquad (8)$$

that is $\Gamma=\Gamma^S P$. We recognize immediately the problem of ''multiplication'' of the random matrix Γ^S by the deterministic projector P. Following the multiplication algorithm [12,20], we construct below the resolvent for the product. First, we construct the resolvent for the projector

$$G_P(z)\equiv\frac{1}{N}\mathrm{Tr}\frac{1}{z-P}=m\frac{1}{z-1}+(1-m)\frac{1}{z}. \qquad (9)$$

The S transform is

$$S_P(z)=\frac{1+z}{m+z}. \qquad (10)$$

The resolvent for the square of the Gaussian is

$$G_{A^2}=\frac{1}{2}\left[1-\left(1-\frac{4}{z}\right)^{1/2}\right]. \qquad (11)$$

Its discontinuity along the real axis reproduces Eq. (7). The corresponding χ and S transformations are

$$\chi_{A^2}=\frac{z}{(1+z)^2}, \quad S_{A^2}=\frac{1}{1+z} \qquad (12)$$

so the product matrix has

$$S=S_{A^2}S_P=\frac{1}{m+z} \qquad (13)$$

and

$$\chi_{A^2\times P}=\frac{z}{(1+z)(m+z)}. \qquad (14)$$

Inverting Eq. (14), and inserting $z(\chi)$ into Eq. (2) we get for the resolvent

$$G(z)=\frac{1-m}{2z}+\frac{1}{2}\left\{1\pm\left[\left(\frac{1-m}{z}-1\right)^2-\frac{4m}{z}\right]^{1/2}\right\}. \qquad (15)$$

The spectral density follows from the discontinuity of Eq. (15) along the real axis in the form

$$\nu(\lambda) = (1-m)\delta(\lambda) + \frac{1}{2\pi}\left[\frac{4m}{\lambda} - \left(\frac{1-m}{\lambda} - 1\right)^2\right]^{1/2}. \quad (16)$$

This result was first obtained in [21], using other methods. The first term, originating from the pole $1/z$ represents $(N-M)$ zero modes of the matrix Γ.

Now we could ''add'' the random Hamiltonian H_R. It is convenient first to solve the Hermitian problem $H_H = H_R + g\Gamma$. This problem is immediately solved using the R transformation (Blue's function), since the resolvent for GOE is known to be

$$G(z) = \tfrac{1}{2}(z - \sqrt{z^2-4}) \quad (17)$$

and the resolvent for Γ is given by Eq. (15). The corresponding Blue's functions are straightforward to find

$$B_1(z) = z + \frac{1}{z},$$

$$B_2(z) = \frac{mg}{1-gz} + \frac{1}{z}, \quad (18)$$

where we have reinstated the coupling g. The labels 1(2) refer to $H_{R(\Gamma)}$, respectively. From the addition law follows the resolvent G of the sum H_H, as a solution to a cubic equation (Cardano class)

$$z = \frac{1}{G(z)} + \frac{mg}{1-gG(z)} + G(z). \quad (19)$$

Out of the three solutions to this equation, we choose uniquely the one respecting the positivity and normalizability of the spectral density. Structural changes in the spectral density can be easily read out from the zeros of $B'(z)=0$ (end points), where B is the Blue's function for the sum. This equation is fourth order (Ferrari class)

$$-1/z^2 + \frac{mg^2}{(1-gz)^2} + 1 = 0 \quad (20)$$

and defines all the end points of the spectrum $A_i = B(z_i)$, $(i=1,2,3,4)$. A structural change (usually a phase transition in the underlying system) happens when two arcs $[A_1,A_2]$ and $[A_3,A_4]$ merge into the one (i.e., $A_2=A_3$). (In the present case, the spectrum is not even in contrast to the chiral cases considered in [22].)

The condition for the structural change (merging point) follows from the zero of the discriminant of the quartic equation (20),

$$[1-(1-m)g^2]^3 + 27g^4m = 0,$$

with the solution

$$g_*^2(m) = \frac{1}{(1-\sqrt[3]{m})^3}. \quad (21)$$

Before closing this section, let us mention that the simpler f case [8], where the matrix elements A_k^a are constrained by

$$\sum_{k=1}^{N} A_k^a A_k^b = \delta^{ab} \quad (a,b=1,\ldots,M) \quad (22)$$

instead of being Gaussian, can be analyzed using similar methods. In a reaction process, the constraint (22) excludes direct reactions. In this case, the eigenvalues of the matrix $g\Gamma$ are either zeros (since A is rectangular) or equal to g [because of Eq. (22)]. The resolvent for Γ in this case has the form of a projector, with a fraction m of eigenvalues equal to g, and the remaining fraction $(1-m)$ equal to zero

$$G_\Gamma(z) = \frac{1-m}{z} + \frac{m}{z-g}. \quad (23)$$

The addition law for the Blue's functions yields the cubic Pastur's equation [23]

$$G_f = \frac{1-m}{z-G_f} + \frac{m}{z-G_f-g}. \quad (24)$$

Repeating the analysis for the end points of the previous example shows that a structural change in the spectrum takes place at

$$g_{*f}^2(m) = [\sqrt[3]{[1-m} + \sqrt[3]{m}]^3. \quad (25)$$

While the first order terms in $\sqrt[3]{m}$ in Eqs. (21) and (25) are equal [8], the remainder is not. Note that Eq. (21) diverges as $m\to 1$, while Eq. (25) is symmetric around $m=0.5$, with g_{*f}^2 varying between 1 and 4.

V. CONFORMAL MAPPING

Under the substitution $g\to ig$ the matrix ensemble becomes complex, taking us to the resonance scattering model. As a result, the eigenvalue distribution is valued in the complex plane. For large z, the resolvent $G(z)$ is a holomorphic function of z, whose form follows from Eq. (19) through $g\to ig$. As z is decreased, $G(z)$ will in general blow up (zero denominator), a signal that the function is no longer holomorphic.

The boundary between the holomorphic and nonholomorphic solutions in the z plane, can be derived very generally using a conformal transformation that maps the cuts of the Hermitian ensemble onto the boundary of its non-Hermitian analog. Indeed, consider the case where a Gaussian random and Hermitian matrix R is added to an arbitrary matrix M. The corresponding Blue's function is

$$B_{R+M}(u) = B_R(u) + B_M(u) - \frac{1}{u} = B_M(u) + u. \quad (26)$$

Substituting $u\to G_M(z)$ we get

$$B_{R+M}[G_M(z)] = z + G_M(z). \quad (27)$$

Let w be a point in the complex plane for which $G_M(z) = G_{R+M}(w)$. Using the definition of Blue's functions, it follows that w is located at

$$w = z + G_M(z). \quad (28)$$

Now, if we were to note that in the *holomorphic* domain, the Blue's function for the Gaussian and non-Hermitian ensemble is $B_{iR} = -z + 1/z$, then

$$B_{R+iR} = B_R + B_{iR} - \frac{1}{z} = \frac{1}{z}. \qquad (29)$$

The anti-Hermitian Gaussian *nullifies* the contribution of the Hermitian Gaussian (the R function in this case is zero). This observation together with Eq. (28) allows for relating a Hermitian resolvent to a non-Hermitian one in the holomorphic domain. Indeed, using Eq. (28) and the present observation, it follows that

$$w = z - 2G_{R+M}(z), \quad G_{iR+M}(w) = G_{R+M}(z) \qquad (30)$$

throughout the holomorphic region. Similarly,

$$w = i[z - 2G_{R+M}(z)], \quad G_{R+iM}(w) = -iG_{R+M}(z) \qquad (31)$$

after a simple rotation of the real and imaginary axes. The transformation (31) maps the cuts of the Hermitian ensemble onto the boundaries between the holomorphic and nonholomorphic regions of the non-Hermitian ensemble, as we now illustrate.

Elliptic distribution. Consider first the simple case where $H = H_R + gH_R$, with H_R being a random Gaussian. Since the Blue's function in the holomorphic region is $B = 1/z + (1+g^2)z$, then the resolvent (inverse) is just

$$z - G = \frac{1}{G} + g^2 G. \qquad (32)$$

The two solutions to Eq. (32) are

$$G(z) = \frac{1}{2(1+g^2)}[z \pm \sqrt{z^2 - 4(1+g^2)}]. \qquad (33)$$

Equations (33) are holomorphic everywhere in the z plane, except on the cut $\mathcal{C} = [-2\sqrt{1+g^2}, 2\sqrt{1+g^2}]$ along the *real* axis. $\mathcal{C}$ is the support for the spectral density associated to H (Wigner's semicircle).

In the holomorphic region, the non-Hermitian case follows by setting g to $i\gamma$ yielding $B = 1/z + (1-\gamma^2)z$. Hence,

$$w - G^a = \frac{1}{G^a} - \gamma^2 G^a \qquad (34)$$

for the resolvent. Using the Hermitian solution (33) and the mapping (30), we can map the Hermitian cut $\mathcal{C}$ onto the boundary delimiting the holomorphic region for Eq. (34), that is

$$w = \frac{1}{1+g^2}\{g^2 z \pm \sqrt{z^2 - 4(1+g^2)}\}, \qquad (35)$$

with $z = t \pm i0$ and t in $\mathcal{C}$. Equation (35) spans an ellipsis with axes $2/\sqrt{1+g^2}$ and $2g^2/\sqrt{1+g^2}$. For $g^2 = 1$ this is just Ginibre's circle. (Here for GUE the scale is set by $\beta = 2$ yielding the radius of the disc to be $\sqrt{2}$. In the case of GOE, where the implicit scale is $\beta = 1$, the radius is 1.)

Resonance scattering model. In the same spirit, the cuts associated to $H_R + g\Gamma$, and given by Eq. (20), can be mapped onto the boundary ∂D delimiting the holomorphic region for the non-Hermitian ensemble $H_R + ig\Gamma$. The mapping (31) requires the solution to Eq. (19) for the Hermitian ensemble. Indeed, after rewriting (19) in terms of w and z, explicitly inserting the form $w = x + iy$, and choosing $z = t$ [where real t scans the cuts of Eq. (19)] one arrives at two coupled equations for x, y, and t. Eliminating t, the equation of the boundary reads

$$x^2 = \frac{4m}{gy} - \left(\frac{g}{1+gy} - \frac{m}{y} - \frac{1}{g}\right)^2 \qquad (36)$$

in agreement with [7].

The results are displayed in Fig. 1 for different couplings g and a fixed ratio $m = 0.25$. The locations of the cuts following from the holomorphic resolvent through Eq. (20) with $g \rightarrow ig$ are found to be strongly correlated with the shape of the envelopes following from the nonholomorphic resolvent. The envelopes can be regarded qualitatively as a smearing of the cuts. For small couplings, say $|g| = 0.1$, the vertical cut lies far on the $x = 0$ axis in the positive y half-plane. It does not show up in our first figure. For larger couplings, say $|g| = 1$, it emerges from the half-plane, and gives rise to a bulge for $|g| = 4$. Above the critical value $|g| = 4.44$ following from Eq. (21), the vertical cut drives the upper island up, towards $y = -\infty$ for $|g| \rightarrow \infty$. At very large couplings, say $|g| = 100$, the islands are indistinguishable from the cuts. The stability of the lower island composed essentially of long-lived states (small width states) is ensured by the zero modes of Γ, as originally suggested by [3,5].

Repeating the calculation above for the f case, using the same transformation (31) with a resolvent G now taken from the solution of Eq. (24), yield the following boundary:

$$x^2 = \frac{4m}{gy} - \left(\frac{1-m}{g-y} - \frac{m}{y} - g\right)^2 \qquad (37)$$

between the holomorphic and nonholomorphic solutions. We note that a similar construction can be used to analyze the non-Hermitian chiral random matrix model [4].

Let us finally mention, that the technique described here does not allow us directly to go *inside* the islands of non-analyticity, where the Green's functions are nonholomorphic. For that we need to generalize the concepts of free random variables for non-Hermitian random matrices as we now discuss.

VI. BLUE'S FUNCTIONS REVISITED

To discuss the generic case of non-Hermitian random matrix models, we need to generalize the concept of R transformation. For that, consider the addition law (1), rewritten in the equivalent way $[z \rightarrow G_{1+2}(z) \equiv G]$,
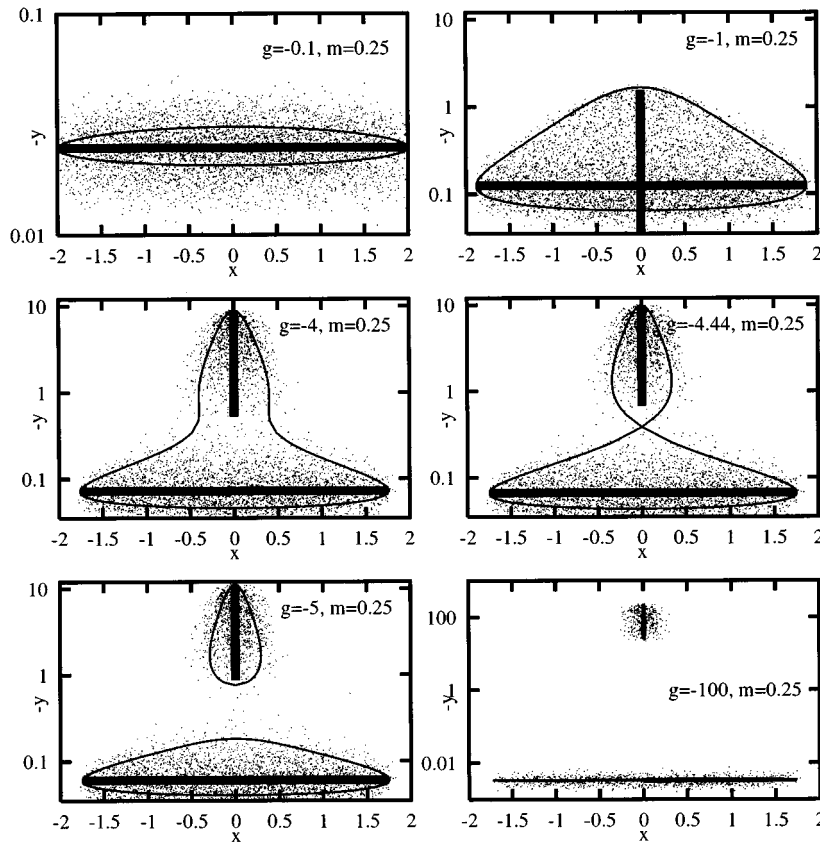
$$z = B_1(G) + B_2(G) - \frac{1}{G}. \qquad (38)$$

FIG. 1. The evolution of the boundary (thin solid line) for various couplings at fixed ratio $m=0.25$. The dots are the numerical eigenvalues generated from an ensemble of 50 matrices that are 100 by 100. The solid bars are the positions of the cuts following from (20) with $g \to ig$.

The generalization to non-Hermitian random matrices amounts to defining 2×2 Green's functions $\mathcal{G}$, and Blue's functions $\mathcal{B}$, such that

$$\mathcal{B}(\mathcal{G}) = \mathcal{Z} = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}. \tag{39}$$

The addition law (R transform) becomes

$$\mathcal{Z} = \mathcal{B}_1(\mathcal{G}) + \mathcal{B}_2(\mathcal{G}) - \frac{1}{\mathcal{G}}. \tag{40}$$

A diagrammatic proof of this relation will be given elsewhere [24]. The present method can be used to analyze the Green's functions of Hermitian and non-Hermitian random matrix models in the entire z plane. We now illustrate this assertion for the two cases discussed above.

Circular ensemble. Consider once more the case $H_R + iH_R$. The generalized Green's function for the Hermitian Gaussian ensemble H_R follows from a straightforward generalization of Pastur's equation [23] (resummation of the rainbow graphs)

$$\mathcal{G} = \frac{1}{\mathcal{Z} - \Sigma} = \Sigma \tag{41}$$

through the substitution $z \to \mathcal{Z}$. Hence,

$$\mathcal{G} + \frac{1}{\mathcal{G}} = \mathcal{Z}. \tag{42}$$

The corresponding Blue's function is

$$\mathcal{B}_R(\mathcal{A}) = \frac{1}{\mathcal{A}} + \mathcal{A}, \tag{43}$$

which is to be compared with Eq. (18). For the non-Hermitian Gaussian ensemble, the *new* version of Pastur's equation is

$$\mathcal{G} \circ \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \Sigma, \tag{44}$$

where the multiplication is meant componentwise. The extra minus sign follows from a sign flip in the "gluon propagator" (non-Hermitian case), while summing over the rainbow graphs [24]. The resulting Blue's function is

$$\mathcal{B}_{iR}(\mathcal{A}) = \frac{1}{\mathcal{A}} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{A} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{45}$$

The Green's function for the sum comes from Eq. (40). Indeed, if we were to define

$$\mathcal{G} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \tag{46}$$

then Eq. (40) together with Eqs. (43) and (45) give

$$\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} = \det \mathcal{G}^{-1} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} + \begin{pmatrix} a & b \\ b & c \end{pmatrix} + \begin{pmatrix} -a & b \\ b & -c \end{pmatrix}. \tag{47}$$

The equation for the off-diagonal element

$$0 = \frac{-b}{\det \mathcal{G}} + 2b \tag{48}$$

has two types of solutions. The case $b = 0$ corresponds to the holomorphic solution $a = G = 1/z$ discussed above, since a in $\mathcal{G}$ is just the resolvent (5). The nonzero case with $\det \mathcal{G} = 1/2$, gives the nonholomorphic (here antiholomorphic) solution $a = G = \overline{z}/2$ in the complementary part of the complex plane, with a uniform eigenvalue distribution $\nu(z, \overline{z}) = 1/2\pi$ as expected from Eq. (6). This is just the result derived by Ginibre and Girko [14].

Random scattering model. Here we consider, for convenience, the ensemble of matrices

$$H' = -gAA^T + iH_R, \tag{49}$$

with again A an $N \times M$ random complex matrix, H_R a Hermitian random matrix. The Hamiltonian H' is related to Hamiltonian (4) by multiplicative factor i, resulting in rotating the axis of the eigenvalue plane by $\pi/2$. The generalized Blue's function for $-gAA^T$ follows from Eq. (18) (second equation) through the substitution $z \to \mathcal{A}$, $g \to -g$. Hence,

$$\mathcal{B}_{gAA^T}(\mathcal{A}) = -\frac{mg}{1 + g\mathcal{A}} + \frac{1}{\mathcal{A}}. \tag{50}$$

Using the addition formula we get

$$\begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} = \frac{-mg}{1 + g\mathcal{G}} + \frac{1}{\mathcal{G}} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{G} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{51}$$

This equation admits a holomorphic and nonholomorphic solution. The former is in agreement with the one discussed above, while the latter reads ($z = x + iy$)

$$G = a = \frac{iy}{2} + \frac{1}{2}\left(-\frac{1}{g} - \frac{m}{x} - \frac{g}{1 - gx} \right) \tag{52}$$

in agreement with [7]. The distribution of eigenvalues has a support only in the nonholomorphic region and follows from Eq. (6). Here, the boundary between the holomorphic and nonholomorphic solution follows from the vanishing of the off-diagonal elements of $\mathcal{G}$ in Eq. (46), reproducing exactly Eq. (36).

Deriving the resolvent for the f case is straightforward, since the problem reduces to finding the solution to the generalized Pastur equation [24]

$$\mathcal{G}_g = \frac{1 - m}{\mathcal{Z} - \widetilde{\mathcal{G}}_f} + \frac{m}{\mathcal{Z} - \widetilde{\mathcal{G}}_f + g}, \tag{53}$$

with

$$\widetilde{\mathcal{G}}_f = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{G}_f \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{54}$$

The nonholomorphic solution for the resolvent (5) reads

$$G_f \equiv a = \frac{iy}{2} + \frac{1}{2}\left(\frac{m - 1}{g + x} - \frac{m}{x} - 2x - g \right) \tag{55}$$

in agreement with [8].

VII. CONCLUSIONS

We have shown how the concepts of addition and multiplication of free random variables could be used to analyze some problems related to the Hermitian and non-Hermitian random matrix models, in the large N limit and for the case of strong non-Hermiticity. For the random scattering model, our level and width distributions are in agreement with those discussed in [7] for strongly overlapping resonances. In this sense, we have not much to add to their general physical discussion.

Our approach extends the work of Voiculescu, Dykema, Nica, and Zee, and to our knowledge, is new. In particular, we have shown that the supports of complex eigenvalues for the non-Hermitian ensemble follow from a simple conformal transformation on the cuts of its Hermitian analog. Our approach offers a simple alternative to the replica and supersymmetric methods for a variety of random matrix models in the large N limit. A comprehensive comparison, including the issue of $1/N$ corrections, is unfortunately beyond the scope of this work.

The resonance scattering model discussed here and the solutions presented above, could be used to discuss some issues related to chaotic behavior in complex systems, such as the instanton liquid model, conductance in quantum dots, microwave cavities, or nuclear dynamics. We hope to address some of these issues next.

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